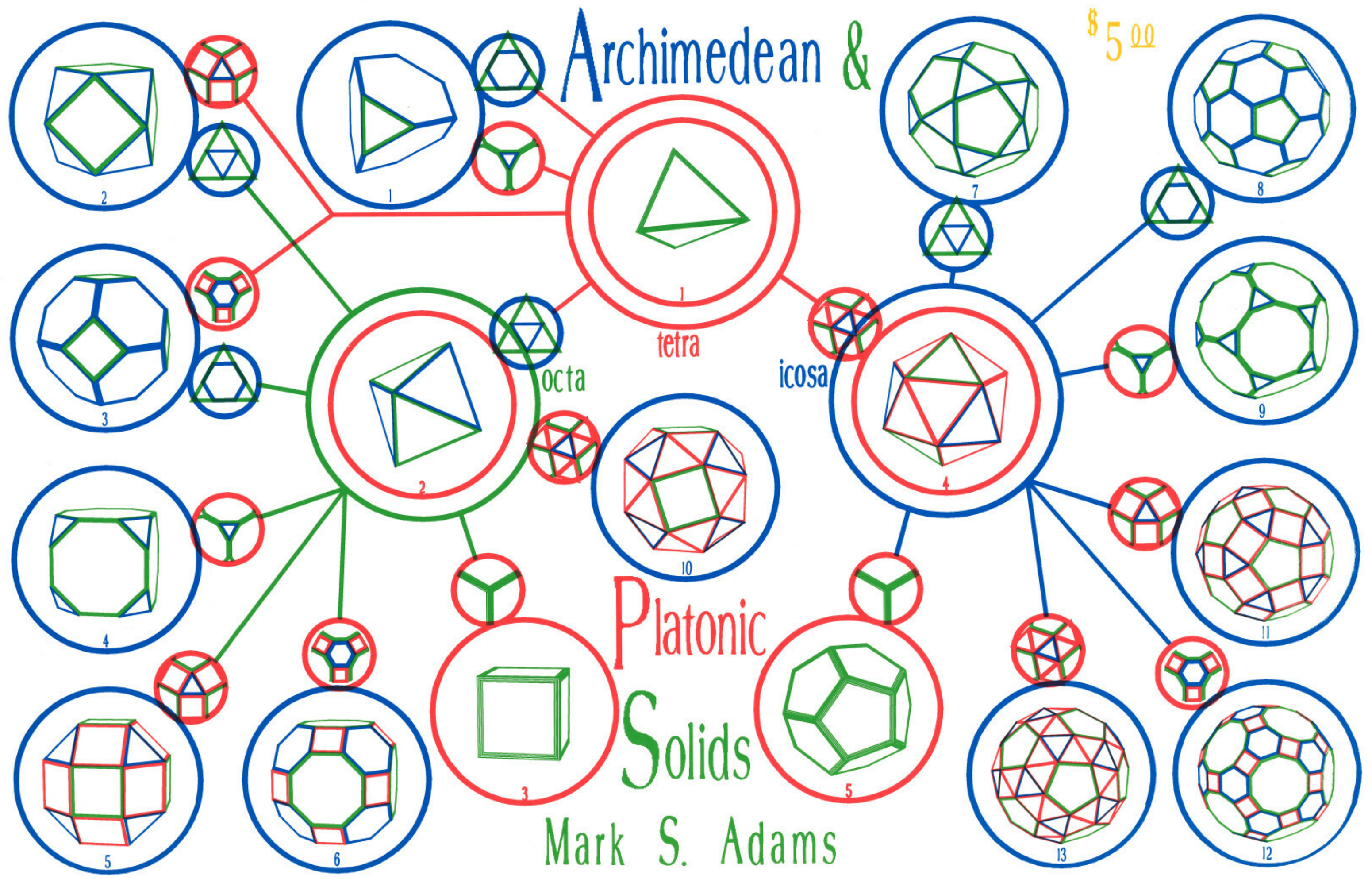


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# Archimedean and Platonic Solids

by Mark S. Adams

*Mark Adams*  
2/26/85

Geodesic Publications

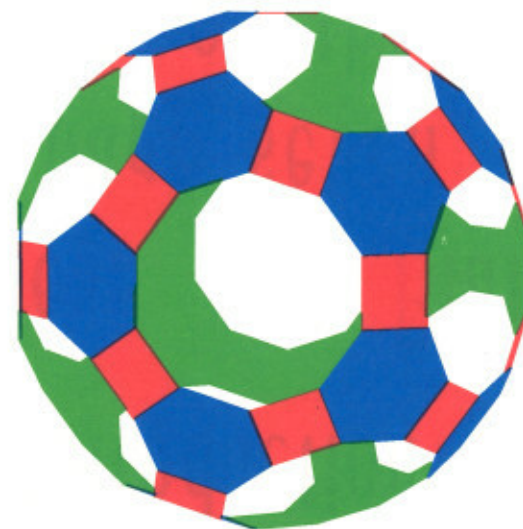
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First Edition.





## Introduction

Volumes of the Archimedean and Platonic solids are presented. Proofs to the solids start on page seven, they are ordered by alternate and triacon geodesic breakdown of the tetrahedron, octahedron, and icosahedron base models. The polyhedron is first inscribed on the face planes of the base model. Its edge distance ( $D$ ) is solved for. Then by dividing the base model radius ( $R$ ) through by ( $D$ ), we have the radius for unit edge length. The Pythagorean theorem is used to project the radius to the center of each face. Summing ( $n$ ) number of volumes of each face pyramid of height ( $r$ ) yields the complete volume.

The fifth frequency triacon has the added freedom of curl angle. ( $\alpha$ ) Positive  $\alpha$  is right-handed, negative  $\alpha$  is left-handed. This breakdown creates the four commensurable volume sets within the eighteen volumes.



Excluding prisms and anti-prisms there exist twenty one semi-regular finite polyhedra, together possessing thirteen distinct volumes. The  $2_v A 5_v T$  and  $5_v T^2$  solids have left and right handed duals. The  $7_v T$ ,  $2_v A 7_v T$ , and  $5_v 7_v T$  solids have interweaving edge rings surrounding each green base model vertex site. Spinnability relocation of red and blue faces around one or more sites forms polarized inter-patterning symmetry. A total of nine realizations of the three solids exist, their volumes are unaffected by the spinnability.

Icosahedral based volumes are plotted showing that powers of the Golden Section ( $\tau$ ) divide alternate and triacon regions. The integer part squared minus radical part squared will be equal to one for even powers of  $\tau$  and minus one for all odd powers. This rule may be extended to the one third power harmonic as plotted.



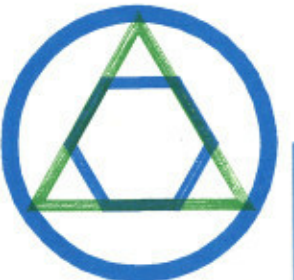

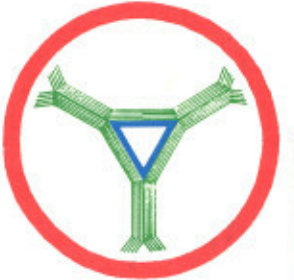
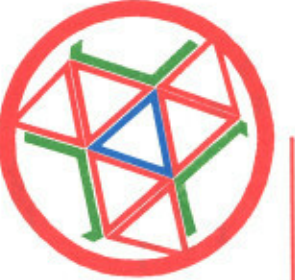
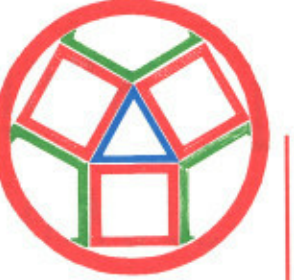


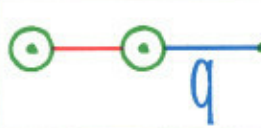
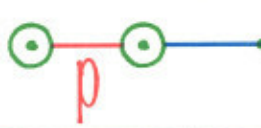
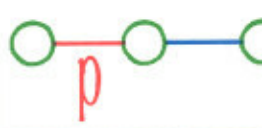
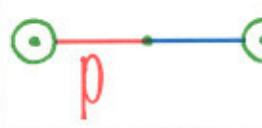
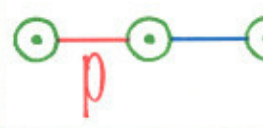

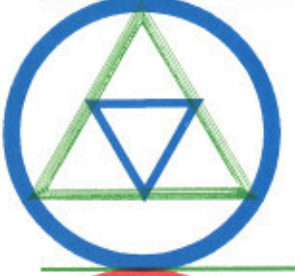



Spherical Tessellations:

regular  $\{p\ q\}$       P Platonic      (o) octa-      (i) ico-      (r) rhombi

quasi-regular  $\{\overset{p}{q}\}$       A Archimedean      (c) cube      (d) dodeca-      (s) snub

     (te) tetra- (hedron)      (co) cubocta-      (id) icosidodeca-      (t) truncated

Wythoff's Construction								
	1v	2vA	3vA	1vT	3vT	5vT	7vT	9vT
								
								
	1vA	2vA	5vT					
	$\{3\ 3\}$	$\{3\ 4\}$	$\{3\ 5\}$	$\{3\ 3\}$	$\{4\ 3\}$	$\{5\ 3\}$	$\{3\ 5\}$	$\{3\ 4\}$
	(te) P1	(o) P2	(i) P4	(te) P1	(c) P3	(d) P5	(co) A2	(t) (o) A3
	$\{3\ 4\}$	$\{3\ 5\}$	$\{5\ 3\}$	$\{4\ 3\}$	$\{5\ 3\}$	$\{5\ 3\}$	$\{3\ 5\}$	$\{3\ 4\}$
	(o) P2	(id) A7	(t) (i) A8	(t) (c) A4	(s) (id) A13	(r) (id) A11	(t) (co) A6	(t) (co) A6
	$\{3\ 5\}$	$\{3\ 5\}$	$\{5\ 3\}$	$\{5\ 3\}$	$\{5\ 3\}$	$\{5\ 3\}$	$\{5\ 3\}$	$\{5\ 3\}$
	(i) P4	(id) A7	(t) (i) A8	(d) P5	(t) (d) A9	(s) (id) A13	(r) (id) A11	(t) (id) A12

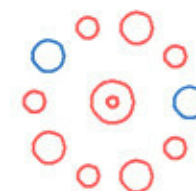
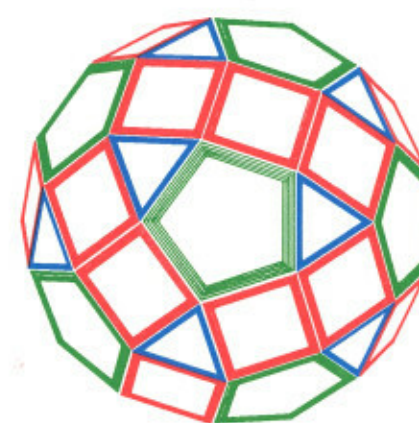
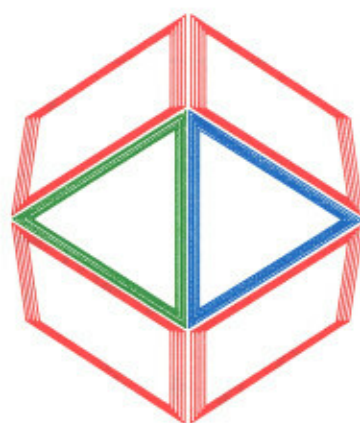
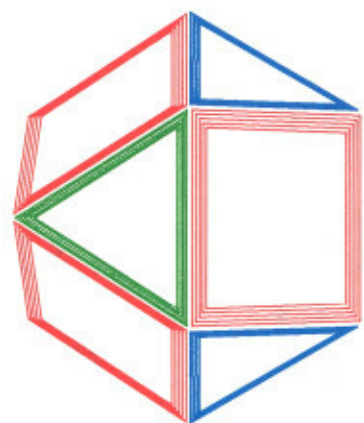


# Alternation of Interpatterning Realizations

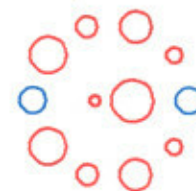
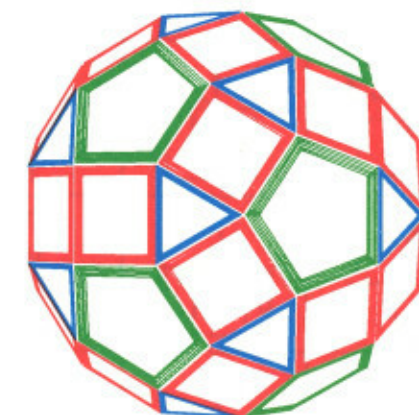
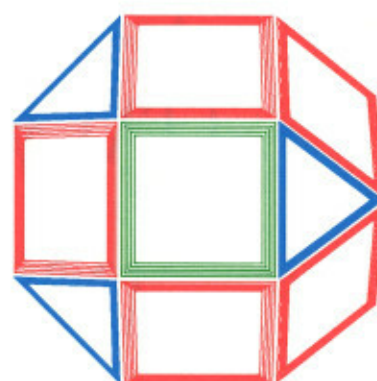
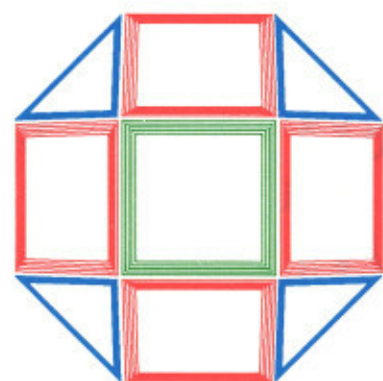
Vector Equilibrium

Vector Polarization

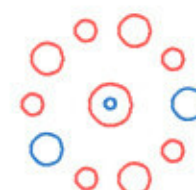
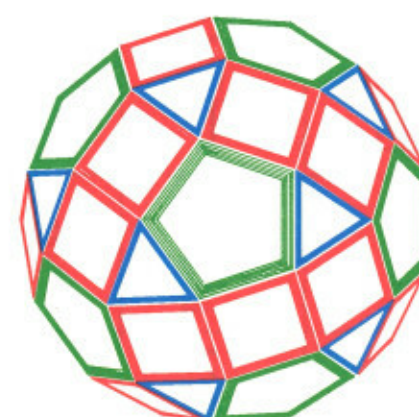
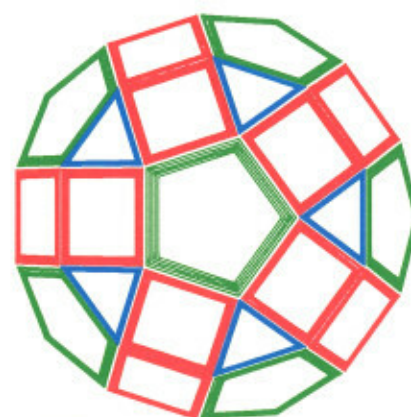
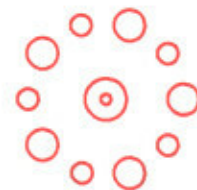
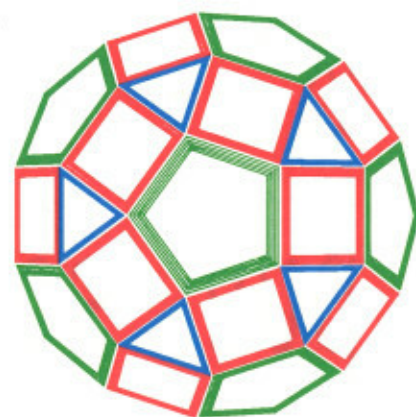
7<sub>v</sub>T



2<sub>v</sub>A7<sub>v</sub>T



5<sub>v</sub>7<sub>v</sub>T





# Volumes and Areas

$$V_{P1} = \frac{1}{6}\sqrt{2}$$

$$V_{P2} = \frac{\sqrt{2}}{3}$$

$$V_{P3} = 1$$

$$V_{P4} = \frac{5\tau^2}{6}$$

$$V_{P5} = \frac{\tau^4\sqrt{5}}{2}$$

$$V_{A1} = \frac{23\sqrt{2}}{12}$$

$$V_{A2} = \frac{5\sqrt{2}}{3}$$

$$V_{A3} = 8\sqrt{2}$$

$$V_{A4} = \frac{(21+14\sqrt{2})}{3}$$

$$V_{A5} = \frac{2}{3}(6+5\sqrt{2})$$

$$V_{A6} = 30+14\sqrt{2}$$

$$V_{A7} = \frac{1}{6}(45+17\sqrt{5})$$

$$V_{A8} = \frac{1}{4}(125+43\sqrt{5})$$







$$V_{A9} = \frac{5}{12}(99+47\sqrt{5})$$

$$V_{A10} = \frac{4}{3}\sqrt{\frac{3}{2}u^2+3u+2} + \sqrt{u\left(\frac{u}{2}+1\right)}$$

$$V_{A11} = \frac{1}{3}(60+29\sqrt{5})$$

$$V_{A12} = 95+50\sqrt{5}$$

$$V_{A13} = \frac{10\tau}{3}\sqrt{\tau^2+3\phi(\tau+\phi)} + \frac{5\tau^2}{2}\sqrt{\frac{1}{5} + \frac{\tau\phi}{\sqrt{5}}(\tau+\phi)}$$

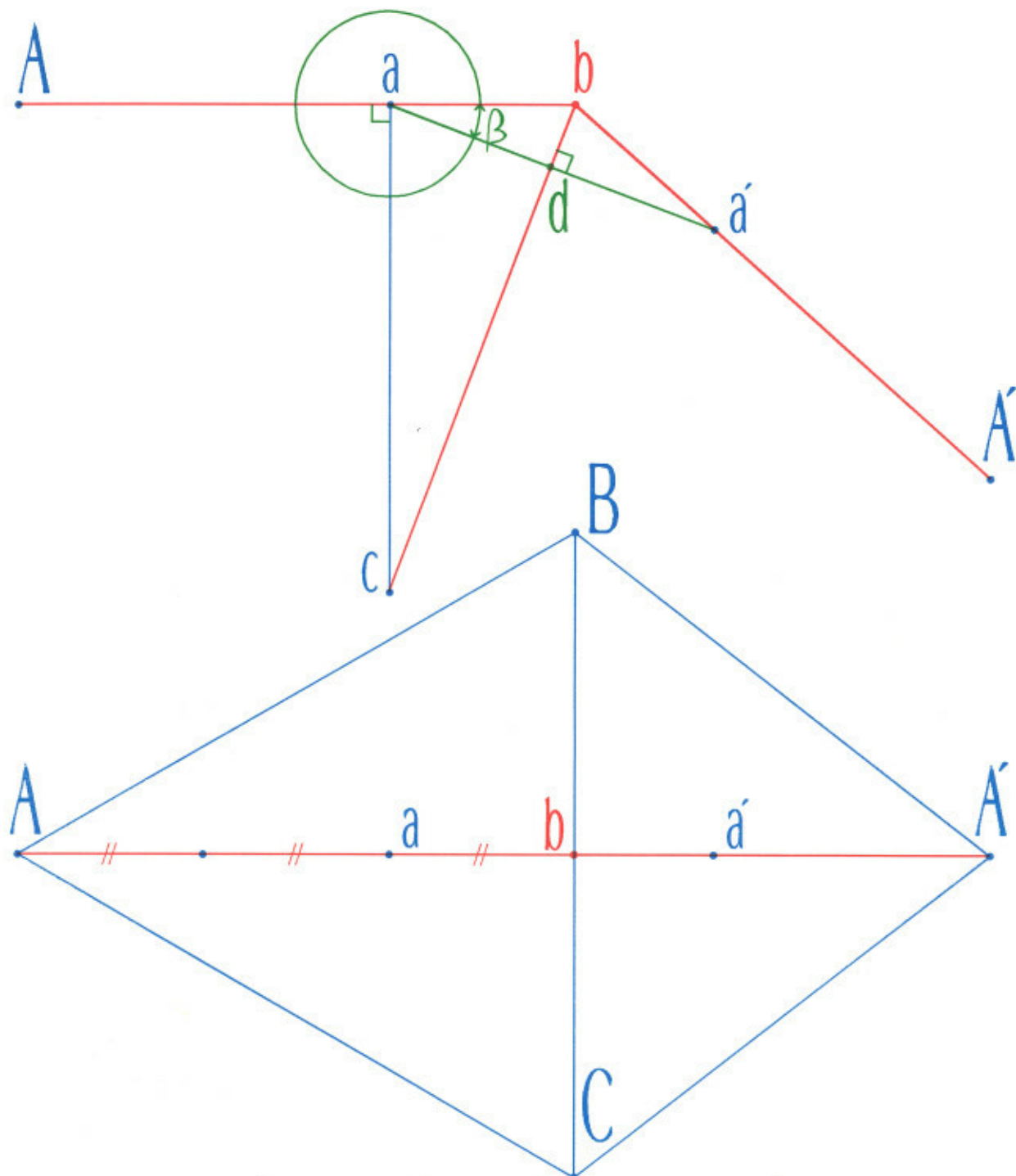
	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$a_{\blacktriangle} = \frac{\sqrt{3}}{4}$
	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$a_{\blacksquare} = 1$
	36°	$\frac{\sqrt{5}}{4\tau}$	$\frac{\tau}{2}$	$\frac{\sqrt{5}}{\tau^3}$	$a_{\blacklozenge} = \frac{5}{4}\sqrt{\frac{\tau^3}{\sqrt{5}}}$
	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{\frac{1}{3}}$	$a_{\bullet} = \frac{3}{2}\sqrt{3}$
	22½°	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{1}{1+\sqrt{2}}$	$a_{\bullet} = 2(1+\sqrt{2})$
	18°	$\frac{1}{2\tau}$	$\frac{\sqrt{\tau\sqrt{5}}}{2}$	$\sqrt{\frac{1}{\tau^3\sqrt{5}}}$	$a_{\ast} = \frac{5}{2}\sqrt{\tau^3\sqrt{5}}$
	$\frac{180^\circ}{s}$	sin	cos	tan	$\frac{s}{4\tan}$

$$\tau \equiv \frac{1+\sqrt{5}}{2}$$

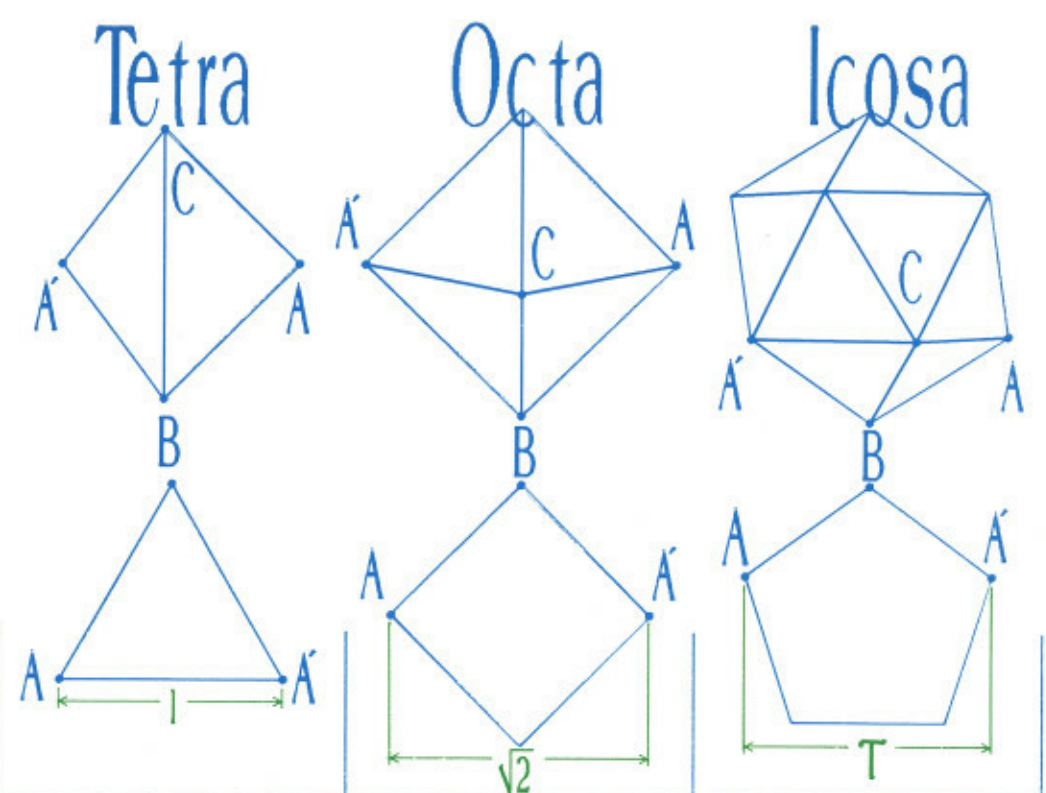
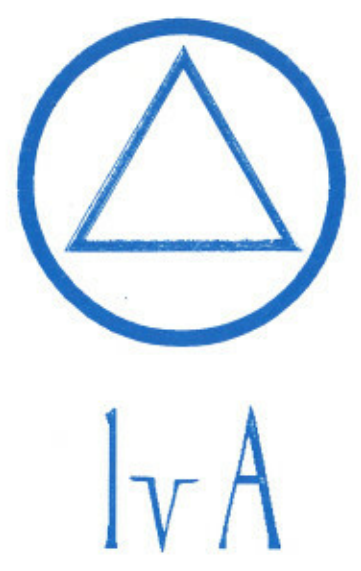
Define **tau** upsilon & phi

$$u \equiv \sqrt[3]{2+\frac{2}{3}\sqrt{\frac{11}{3}}} + \sqrt[3]{2-\frac{2}{3}\sqrt{\frac{11}{3}}} \quad \phi \equiv \sqrt[3]{\frac{\tau}{2}+\sqrt{\frac{\tau^2}{4}-\frac{8}{27}}} + \sqrt[3]{\frac{\tau}{2}-\sqrt{\frac{\tau^2}{4}-\frac{8}{27}}}$$



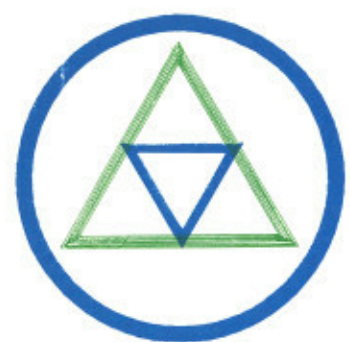


a: center of $\triangle ABC$	a': center of $\triangle A'CB$
$AB=AC=BC=BA'=CA'=1$	b: bisector of $BC$
c: center of solid	d: $aa'$ intersects $bc$

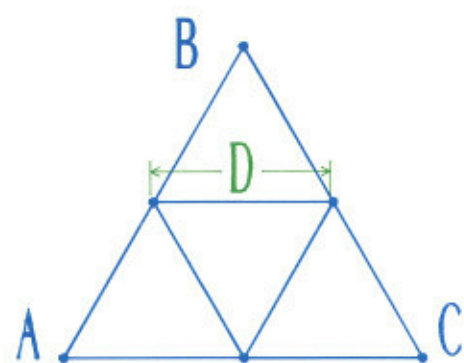


$\overline{ad} = \overline{AA'}/6$	$\frac{1}{6}$	$\frac{\sqrt{2}}{6}$	$\frac{\tau}{6}$
$\cos \beta = \overline{ad} / \overline{ab}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$	$\frac{\tau}{\sqrt{3}}$
$\sin \beta = \sqrt{1 - \cos^2 \beta}$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	$\frac{1}{\tau \sqrt{3}}$
$\overline{bc} = \overline{ab} / \sin \beta$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2}$	$\frac{\tau}{2}$
$\overline{ac} = \overline{bc} \cos \beta$	$R_{T\blacktriangle} = \frac{1}{2\sqrt{6}}$	$R_{O\blacktriangle} = \frac{1}{\sqrt{6}}$	$R_{I\blacktriangle} = \frac{\tau^2}{2\sqrt{3}}$
$n_{\blacktriangle}$	4	8	20
$n_{\blacktriangle} a_{\blacktriangle} \frac{1}{3} R_{\blacktriangle}$	$V_{I_v A} = \frac{1}{6\sqrt{2}}$	$V_{2_v A} = \frac{\sqrt{2}}{3}$	$V_{5_v \tau} = \frac{5\tau^2}{6}$





$2vA$



$$D = \frac{1}{2}$$

$$r_{\blacktriangle} = \frac{R_{\blacktriangle}}{D} = 2R_{\blacktriangle}$$

Tetra:

$$V_{2vA} = \frac{\sqrt{2}}{3}$$

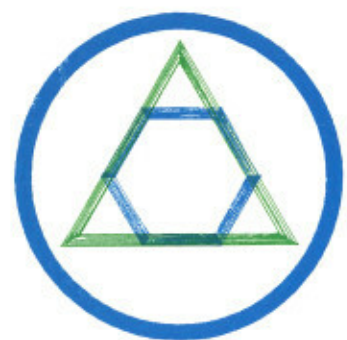
Octa:  $r_{\blacktriangle} = 2R_{\circ\blacktriangle} = \sqrt{\frac{2}{3}}$   $r_{\blacksquare} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60^\circ)^2 - (2\tan 45^\circ)^2} = \sqrt{\frac{8+1-3}{12}} = \frac{1}{\sqrt{2}}$

$$V_{2vA^2} = n_{\blacktriangle} a_{\blacktriangle} \frac{1}{3} r_{\blacktriangle} + n_{\blacksquare} a_{\blacksquare} \frac{1}{3} r_{\blacksquare} = 8 \frac{\sqrt{3}}{4} \frac{1}{3} \sqrt{\frac{2}{3}} + 6 \cdot 1 \cdot \frac{1}{3} \frac{1}{\sqrt{2}} = \frac{5\sqrt{2}}{3}$$

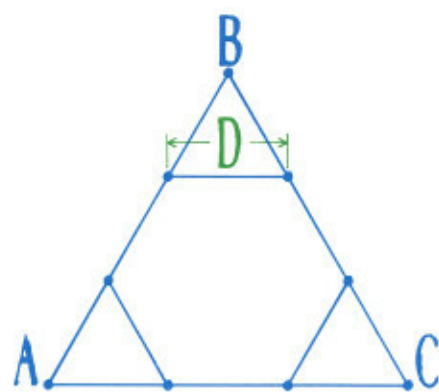
Icosa:  $r_{\blacktriangle} = 2R_{\text{I}\blacktriangle} = \frac{\sqrt{3}}{3}$   $r_{\blacklozenge} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60^\circ)^2 - (2\tan 36^\circ)^2} = \sqrt{\frac{10(7+3\sqrt{5})+5-3\sqrt{5}(2+\sqrt{5})}{60}} = \sqrt{\frac{\sqrt{5}}{5}}$

$$V_{5v\text{I}2vA} = n_{\blacktriangle} a_{\blacktriangle} \frac{1}{3} r_{\blacktriangle} + n_{\blacklozenge} a_{\blacklozenge} \frac{1}{3} r_{\blacklozenge} = 20 \frac{\sqrt{3}}{4} \frac{1}{3} \frac{\sqrt{3}}{3} + 12 \frac{5}{4} \sqrt{\frac{\sqrt{5}}{5}} \frac{1}{3} \sqrt{\frac{\sqrt{5}}{5}} = \frac{45+17\sqrt{5}}{6}$$





3vA



$$D = \frac{1}{3}$$

$$r_{\bullet} = \frac{R_{\blacktriangle}}{D} = 3R_{\blacktriangle}$$

$$\text{Tetra: } r_{\bullet} = 3R_{T\blacktriangle} = \sqrt{\frac{3}{8}} \quad r_{\blacktriangle} = \sqrt{r_{\bullet}^2 + (2 \tan 30^\circ)^2 - (2 \tan 60^\circ)^2} = \sqrt{\frac{9+18-2}{24}} = \frac{5}{2\sqrt{6}}$$

$$V_{3vA} = n_{\bullet} a_{\bullet} \frac{1}{3} r_{\bullet} + n_{\blacktriangle} a_{\blacktriangle} \frac{1}{3} r_{\blacktriangle} = 4 \frac{3\sqrt{3}}{2} \frac{1}{3} \sqrt{\frac{3}{8}} + 4 \frac{\sqrt{3}}{4} \frac{1}{3} \frac{5}{2\sqrt{6}} = \frac{23\sqrt{2}}{12}$$

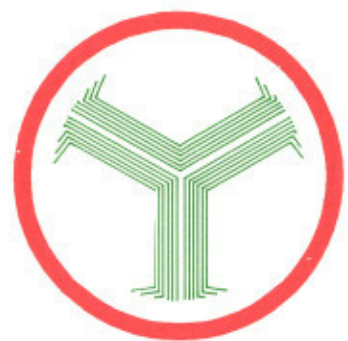
$$\text{Octa: } r_{\bullet} = 3R_{O\blacktriangle} = \sqrt{\frac{3}{2}} \quad r_{\blacksquare} = \sqrt{r_{\bullet}^2 + (2 \tan 30^\circ)^2 - (2 \tan 45^\circ)^2} = \sqrt{\frac{6+3-1}{4}} = \sqrt{2}$$

$$V_{2v3vA} = n_{\bullet} a_{\bullet} \frac{1}{3} r_{\bullet} + n_{\blacksquare} a_{\blacksquare} \frac{1}{3} r_{\blacksquare} = 8 \frac{3\sqrt{3}}{2} \frac{1}{3} \sqrt{\frac{3}{2}} + 6 \cdot 1 \cdot \frac{1}{3} \sqrt{2} = 8\sqrt{2}$$

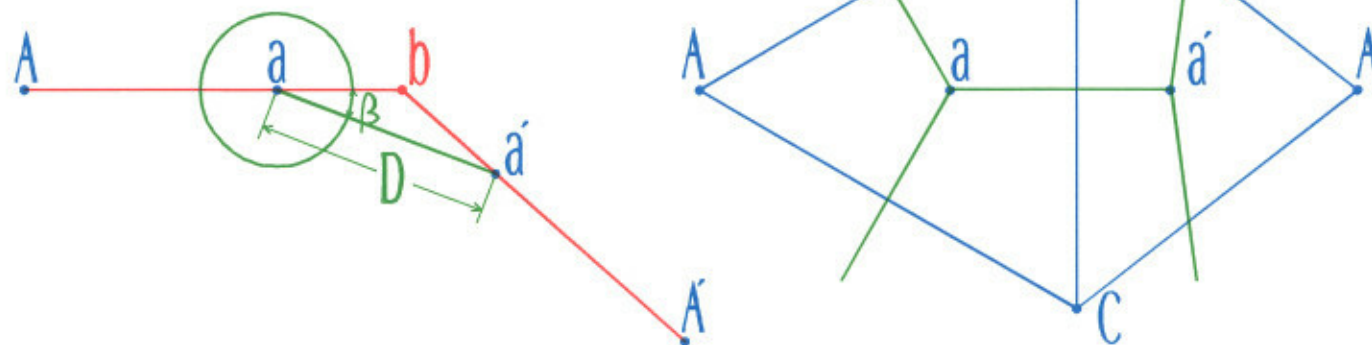
$$\text{Icosa: } r_{\bullet} = 3R_{I\blacktriangle} = \frac{\tau^2 \sqrt{3}}{2} \quad r_{\blacklozenge} = \sqrt{r_{\bullet}^2 + (2 \tan 30^\circ)^2 - (2 \tan 36^\circ)^2} = \sqrt{\frac{15(7+3\sqrt{5})+30-10+4\sqrt{5}}{40}} = \sqrt{\frac{41+25\sqrt{5}}{8\sqrt{5}}}$$

$$V_{5vT3vA} = n_{\bullet} a_{\bullet} \frac{1}{3} r_{\bullet} + n_{\blacklozenge} a_{\blacklozenge} \frac{1}{3} r_{\blacklozenge} = 20 \frac{3\sqrt{3}}{2} \frac{1}{3} \frac{\tau^2 \sqrt{3}}{2} + 12 \frac{5}{4} \sqrt{\frac{\tau^3}{5}} \frac{1}{3} \sqrt{\frac{41+25\sqrt{5}}{8\sqrt{5}}} = \frac{125+43\sqrt{5}}{4}$$





$lvT$



$$D = \overline{aa'} = 2 ab \cos \beta = \frac{\cos \beta}{\sqrt{3}}$$

Vertex Radius:

$$r_v = \frac{R_{\blacktriangle}}{D} = \frac{R_{\blacktriangle} \sqrt{3}}{\cos \beta}$$

Octa:

$$r_v = \frac{R_{\blacktriangle} \sqrt{3}}{\cos \beta_0} = \frac{\sqrt{3}}{2}$$

$$r_{\blacksquare} = \sqrt{r_v^2 - (2 \sin 45^\circ)^{-2}} = \sqrt{\frac{3-2}{4}} = \frac{1}{2}$$

$$V_{2vA} lvT = n_{\blacksquare} a_{\blacksquare} \frac{1}{3} r_{\blacksquare} = 6 \mid \frac{1}{3} \frac{1}{2} = 1$$

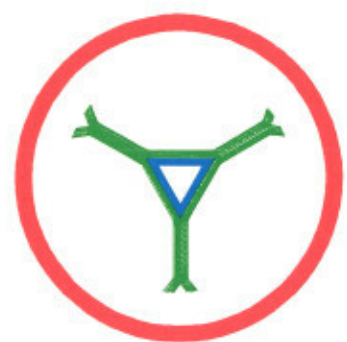
Icosa:

$$r_v = \frac{R_{\blacktriangle} \sqrt{3}}{\cos \beta_1} = \frac{\tau \sqrt{3}}{2}$$

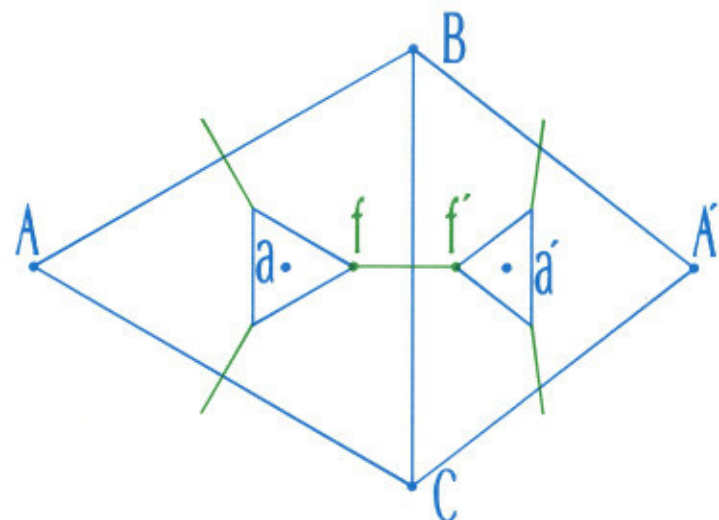
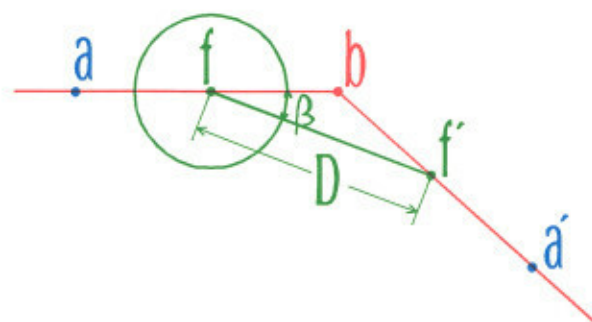
$$r_{\blacklozenge} = \sqrt{r_v^2 - (2 \sin 36^\circ)^{-2}} = \sqrt{\frac{15(3+\sqrt{5}) - 4\sqrt{5}(1+\sqrt{5})}{40}} = \sqrt{\frac{\tau^5}{4\sqrt{5}}}$$

$$V_{5v} lvT = n_{\blacklozenge} a_{\blacklozenge} \frac{1}{3} r_{\blacklozenge} = 12 \frac{5}{4} \sqrt{\frac{\tau^3}{5}} \frac{1}{3} \sqrt{\frac{\tau^5}{4\sqrt{5}}} = \frac{\tau^4 \sqrt{5}}{2}$$





3vT



$$\overline{ab} = \overline{af} + \overline{fb}$$

or

$$\frac{1}{2\sqrt{3}} = D \left( \frac{1}{\sqrt{3}} + \frac{1}{2\cos\beta} \right)$$

$$r_{\blacktriangle} = \frac{R_{\blacktriangle}}{D} = R_{\blacktriangle} \left( 2 + \frac{\sqrt{3}}{\cos\beta} \right)$$

Octa:

$$r_{\blacktriangle} = R_{\blacktriangle} \left( 2 + \frac{\sqrt{3}}{\cos\beta_0} \right) = \frac{3+2\sqrt{2}}{2\sqrt{3}}$$

$$r_{\bullet} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60^\circ)^2 - (2\tan 22\frac{1}{2}^\circ)^2} = \sqrt{\frac{17+12\sqrt{2} + 1 - 3(3+2\sqrt{2})}{12}} = \frac{1+\sqrt{2}}{2}$$

$$V_{2vA3vT} = n_{\blacktriangle} a_{\blacktriangle} \frac{1}{3} r_{\blacktriangle} + n_{\bullet} a_{\bullet} \frac{1}{3} r_{\bullet} = 8 \frac{\sqrt{3}}{4} \frac{1}{3} \frac{3+2\sqrt{2}}{2\sqrt{3}} + 6 \frac{2(1+\sqrt{2})}{3} \frac{1}{3} \frac{1+\sqrt{2}}{2} = \frac{7}{3}(3+2\sqrt{2})$$

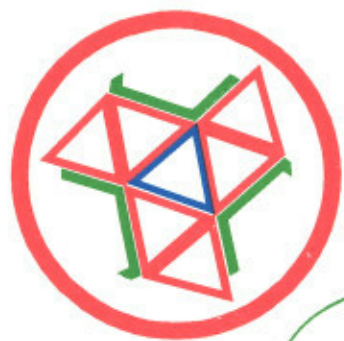
Icosa:

$$r_{\blacktriangle} = R_{\blacktriangle} \left( 2 + \frac{\sqrt{3}}{\cos\beta_1} \right) = \frac{9+5\sqrt{5}}{4\sqrt{3}}$$

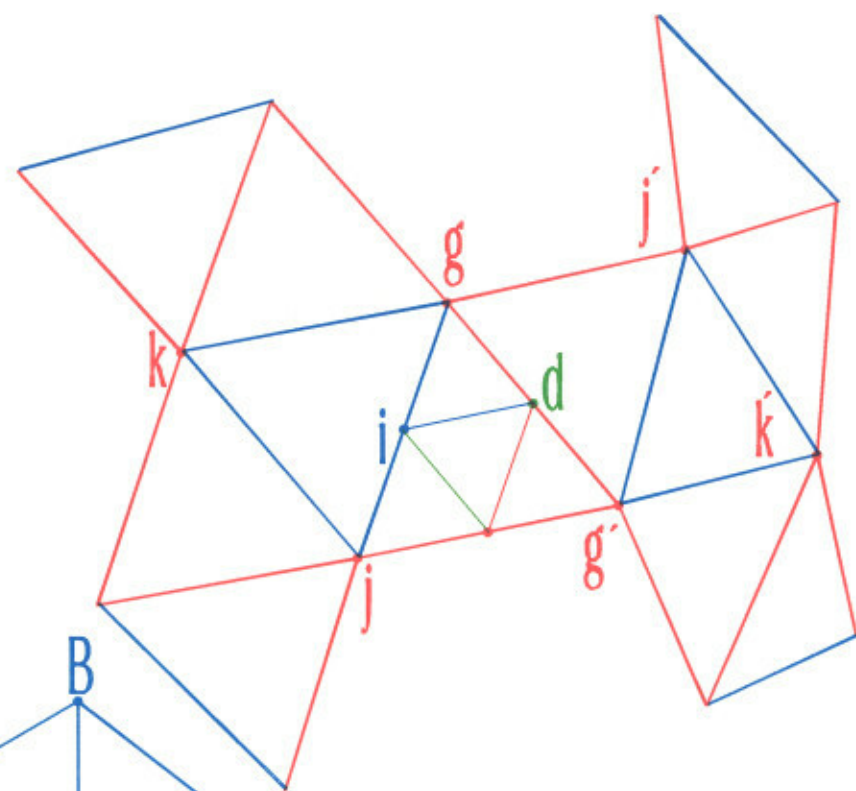
$$r_{\blackstar} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60^\circ)^2 - (2\tan 18^\circ)^2} = \sqrt{\frac{(9+5\sqrt{5})^2 + 4 - 12\sqrt{5}(2+\sqrt{5})}{48}} = \frac{1}{2}\sqrt{7^5\sqrt{5}}$$

$$V_{5v3vT} = n_{\blacktriangle} a_{\blacktriangle} \frac{1}{3} r_{\blacktriangle} + n_{\blackstar} a_{\blackstar} \frac{1}{3} r_{\blackstar} = 20 \frac{\sqrt{3}}{4} \frac{1}{3} \frac{9+5\sqrt{5}}{4\sqrt{3}} + 12 \frac{5\sqrt{7^3\sqrt{5}}}{2} \frac{1}{3} \frac{1}{2}\sqrt{7^5\sqrt{5}} = \frac{5}{12}(99+47\sqrt{5})$$





# Fifth Frequency Triacon:



$$\overline{gj} = \overline{jk} = \overline{kg} = \overline{gg'} = \overline{g'j} = D$$

$$\overline{gi} = \overline{ij} = \overline{id} = \overline{gd} = \overline{dg'} = \frac{D}{2}$$

$$\overline{af}^2 + \overline{fg}^2 = \overline{ag}^2 = \frac{D^2}{3}$$

$$\overline{ah}^2 + \overline{hi}^2 = \overline{ai}^2 = \frac{D^2}{12}$$

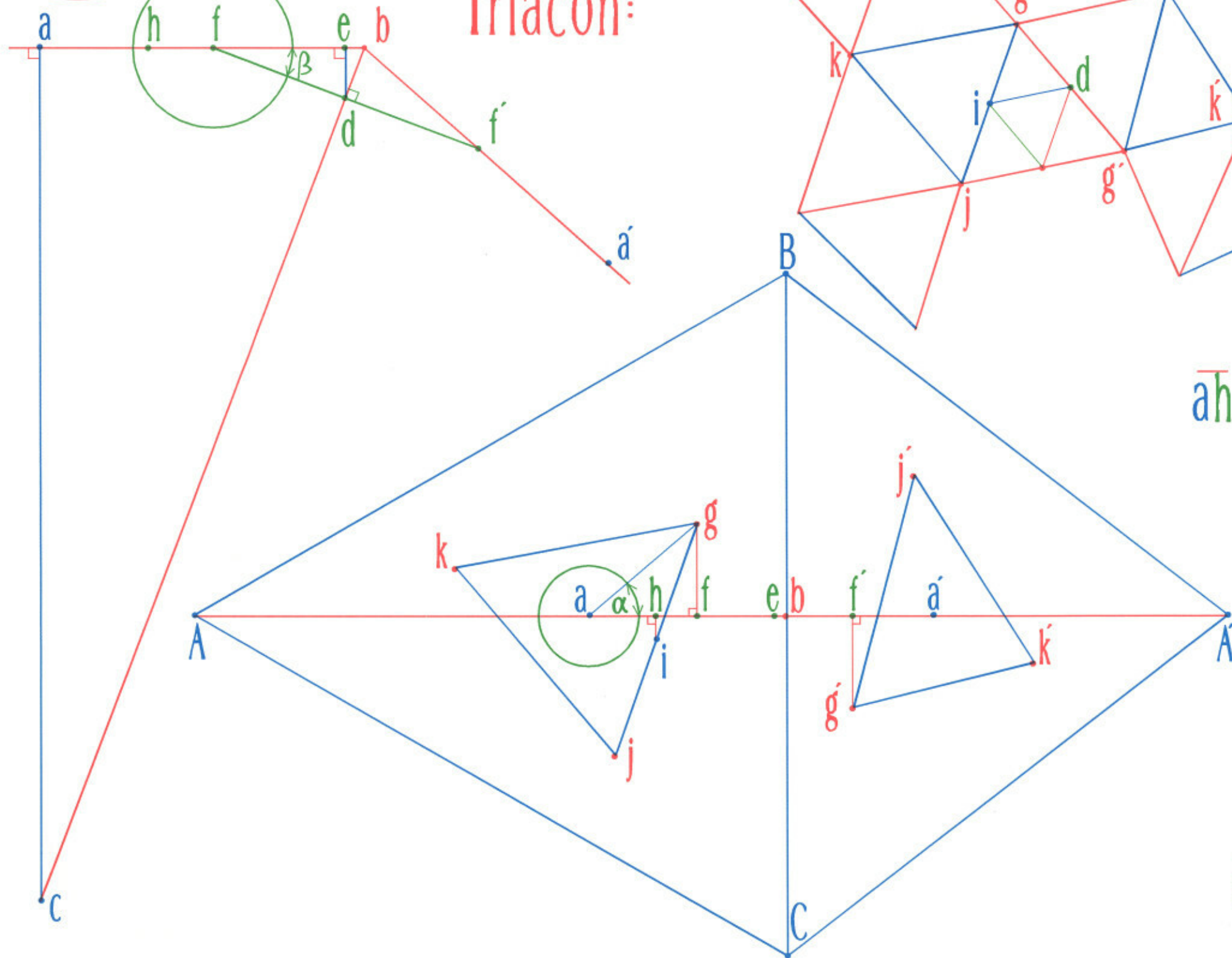
$$\overline{ah} = \overline{ai} \cos(60 - \alpha) = \frac{D}{4\sqrt{3}} (\cos \alpha + \sqrt{3} \sin \alpha)$$

$$\overline{af} = \overline{ag} \cos \alpha = \frac{D}{\sqrt{3}} \cos \alpha$$

$$\overline{fb} = \overline{ab} - \overline{af} = \frac{1}{2\sqrt{3}} (1 - 2D \cos \alpha)$$

$$\overline{eb} = \overline{db} \sin \beta = \overline{fb} \sin^2 \beta$$

$$\overline{eb}^2 + \overline{ed}^2 = \overline{eb}^2 (1 + \cot^2 \beta) = \overline{eb} \overline{fb}$$





$$\begin{aligned}
\overline{gd}^2 - \frac{D^2}{4} = 0 &= \overline{gf}^2 + (\overline{ab} - \overline{af} - \overline{eb})^2 + \overline{ed}^2 - \frac{D^2}{4} \\
&= \overline{ag}^2 + \overline{ab}^2 - 2\overline{ab}\overline{af} + \overline{eb}[2\overline{af} - 2\overline{ab} + \overline{fb}] - \frac{D^2}{4} \\
&= \frac{D^2}{3} + \frac{1}{12} - \frac{D}{3}\cos\alpha + \sin^2\beta \frac{1-2D\cos\alpha}{2\sqrt{3}} \left[ \frac{2D\cos\alpha}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{1-2D\cos\alpha}{2\sqrt{3}} \right] - \frac{D^2}{4}
\end{aligned}$$

or,  $D^2 - 4D\cos\alpha + 1 - \sin^2\beta (1 - 2D\cos\alpha)^2 = 0$  Eq. 1

$$\begin{aligned}
\overline{id}^2 - \frac{D^2}{4} = 0 &= \overline{hi}^2 + (\overline{ab} - \overline{ah} - \overline{eb})^2 + \overline{ed}^2 - \frac{D^2}{4} \\
&= \overline{ai}^2 + \overline{ab}^2 - 2\overline{ab}\overline{ah} + \overline{eb}[2\overline{ah} - 2\overline{ab} + \overline{fb}] - \frac{D^2}{4} \\
&= \frac{D^2}{12} + \frac{1}{12} - \frac{D}{12}(\cos\alpha + \sqrt{3}\sin\alpha) + \sin^2\beta \frac{1-2D\cos\alpha}{2\sqrt{3}} \left[ \frac{D(\cos\alpha + \sqrt{3}\sin\alpha)}{2\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{1-2D\cos\alpha}{2\sqrt{3}} \right] - \frac{D^2}{4}
\end{aligned}$$

or,  $-2D^2 - D(\cos\alpha + \sqrt{3}\sin\alpha) + 1 - \sin^2\beta (1 - 2D\cos\alpha)[1 + D(\cos\alpha - \sqrt{3}\sin\alpha)] = 0$  Eq. 2



Define Gamma Operators:  $\gamma \equiv \sqrt{3} \tan \alpha$      $\Gamma \equiv 3 \cos \alpha - \sqrt{3} \sin \alpha$

$$\cos^2 \alpha + \sin^2 \alpha = \cos^2 \alpha \left( 1 + \frac{\gamma^2}{3} \right) = 1 \quad \text{so,} \quad \cos^2 \alpha = \frac{1}{1 + \frac{\gamma^2}{3}}$$

$$\Gamma \cos \alpha = (3 \cos \alpha - \sqrt{3} \sin \alpha) \cos \alpha = (3 - \gamma) \cos^2 \alpha = \frac{3 - \gamma}{1 + \frac{\gamma^2}{3}}$$

$$3 \left( 1 + \frac{\gamma^2}{3} \right) \left[ \Gamma \cos \alpha - \frac{3 - \gamma}{1 + \frac{\gamma^2}{3}} \right] = \underbrace{\Gamma \cos \alpha}_{a} \gamma^2 + \underbrace{3 \gamma}_{b} + \underbrace{3(\Gamma \cos \alpha - 3)}_{c} = 0$$

Positive Root of  $\gamma$ :  $\gamma = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-3 + \sqrt{9 - 12 \Gamma \cos \alpha (\Gamma \cos \alpha - 3)}}{2 \Gamma \cos \alpha}$  Eq. 3

$$\Gamma^2 = \Gamma \cos \alpha (3 - \gamma) = 3 \Gamma \cos \alpha - \frac{1}{2} \left[ -3 + \sqrt{9 - 12 \Gamma \cos \alpha (\Gamma \cos \alpha - 3)} \right] \quad \underline{\underline{\text{Eq. 4}}}$$



Octa: Eq.1:  $(3 - 4\cos^2\alpha)D^2 - 8\cos\alpha D + 2 = F_{01} = 0$

$\sin^2\beta_0 = \frac{1}{3}$  Eq.2:  $2(\cos\alpha - \sqrt{3}\sin\alpha)D^2 - 2(\cos\alpha + \sqrt{3}\sin\alpha)D + 2 = F_{02} = 0$

$\frac{F_{02} - F_{01}}{D} = (2(3\cos\alpha - \sqrt{3}\sin\alpha)\cos\alpha - 9)D + 2(3\cos\alpha - \sqrt{3}\sin\alpha) = (2\Gamma\cos\alpha - 9)D + 2\Gamma = 0$  so,  $D = \frac{2\Gamma}{9 - 2\Gamma\cos\alpha}$  Eq.5

$\frac{\Gamma(9 - 2\Gamma\cos\alpha)}{3D} F_{01} = \frac{\Gamma(9 - 2\Gamma\cos\alpha)}{3} \left[ (3 - 4\cos^2\alpha) \left( \frac{2\Gamma}{9 - 2\Gamma\cos\alpha} \right) - 8\cos\alpha + 2 \left( \frac{9 - 2\Gamma\cos\alpha}{2\Gamma} \right) \right] = 0$

$= 4(\Gamma\cos\alpha)^2 - 36\Gamma\cos\alpha + 27 + 2\Gamma^2 = 0$  from Eq.4:

$4(\Gamma\cos\alpha)^2 - 36\Gamma\cos\alpha + 27 + 6\Gamma\cos\alpha + 3 = \sqrt{9 - 12\Gamma\cos\alpha(\Gamma\cos\alpha - 3)}$

Square both sides and subtract:

$16(\Gamma\cos\alpha)^4 - 240(\Gamma\cos\alpha)^3 + 1152(\Gamma\cos\alpha)^2 - 1836\Gamma\cos\alpha + 891 = 0$

Define:  $x = \frac{2}{3}\Gamma\cos\alpha$  and divide by 81:  $x^4 - 10x^3 + 32x^2 - 34x + 11 = 0$



First Root of  $x$ :  $x_{01} = 1$

Second Root:  $a = q - \frac{p^2}{3} = 23 - 27 = -4$

$$b = \frac{2p^3}{27} - \frac{pq}{3} + r = -54 + 69 - 11 = 4$$

$$x_{02} = -\frac{p}{3} - \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} - \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} = 3 - v$$

$$(x-1) \left[ \begin{array}{r} x^3 - \overset{p}{9}x^2 + \overset{q}{23}x - \overset{r}{11} \\ \hline x^4 - 10x^3 + 32x^2 - 34x + 11 \\ \hline x^4 - x^3 \\ \hline 0 - 9x^3 + 9x^2 \\ \hline 0 \quad 23x^2 - 23x \\ \hline 0 \quad -11x + 11 \\ \hline 0 \quad 0 \end{array} \right]$$

From the second root,  $\Gamma = \frac{3}{2}(3-v)$  Eq. 6

note:  $\left(\sqrt[3]{2 + \frac{2}{3}\sqrt{11}}\right) \left(\sqrt[3]{2 - \frac{2}{3}\sqrt{11}}\right) = \sqrt[3]{\frac{64}{27}} = \frac{4}{3}$  therefore,  $v^3 = 4(v+1)$  Eq. 7

also note:  $(20 + 6v - 3v^2)^2 - [(3v + 2)^2(-3v^2 + 12v - 8)] = 0$

$$= 400 + 36v^2 + 36v(v+1) + 240v - 120v^2 - 144(v+1) - [-108v(v+1) - 144(v+1) - 12v^2 + 432(v+1) + 144v^2 + 48v - 72v^2 - 96v - 32] = 0$$

$$\text{therefore: } \sqrt{-3v^2 + 12v - 8} = \frac{20 + 6v - 3v^2}{3v + 2} \quad \underline{\underline{\text{Eq. 8}}}$$



$$\begin{aligned} \gamma &= \frac{-3 + \sqrt{9 - 12\Gamma \cos \alpha (\Gamma \cos \alpha - 3)}}{2\Gamma \cos \alpha} = \frac{-1 + \sqrt{-3v^2 + 12v - 8}}{3 - v} = \frac{-(3v+2) + (20+6v-3v^2)}{(3-v)(3v+2)} = \frac{v+2}{v+\cancel{2}/3} \\ \text{Eq. 3} & \qquad \qquad \qquad \text{Eq. 6} & \qquad \qquad \qquad \text{Eq. 8} \end{aligned}$$

$$\cos \alpha = \frac{1}{\sqrt{1 + \tan^2 \alpha}} = \frac{1}{\sqrt{1 + \frac{\gamma^2}{3}}} = \frac{1}{\sqrt{1 + \frac{1}{3} \left( \frac{v+2}{v+\cancel{2}/3} \right)^2}} = \frac{3v+2}{2\sqrt{3v^2+6v+4}}$$

$$\Gamma = 3\cos \alpha - \sqrt{3} \sin \alpha = \cos \alpha [3 - \gamma] = \frac{(3v+2)}{2\sqrt{3v^2+6v+4}} \left[ \frac{3(3v+2) - 3(v+2)}{3v+2} \right] = \frac{3v}{\sqrt{3v^2+6v+4}}$$

$$\begin{aligned} \text{Eq. 5: } D &= \frac{2\Gamma}{9 - 2\Gamma \cos \alpha} = \frac{2\Gamma}{3v} = \frac{2}{\sqrt{3v^2+6v+4}} & r_{\blacktriangle} &= \frac{R_{0\blacktriangle}}{D} = \frac{\sqrt{3v^2+6v+4}}{2\sqrt{6}} \\ & \qquad \qquad \qquad \text{Eq. 6} \end{aligned}$$

$$r_{\blacksquare} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60)^{-2} - (2\tan 45)^{-2}} = \sqrt{\frac{1}{24}(3v^2+6v+4+2-6)} = \frac{1}{2}\sqrt{v\left(\frac{v}{2}+1\right)}$$

$$\begin{aligned} V_{2vA5vT} &= n_{\blacktriangle} a_{\blacktriangle} \frac{1}{3} r_{\blacktriangle} + n_{\blacksquare} a_{\blacksquare} \frac{1}{3} r_{\blacksquare} = 32 \frac{\sqrt{3}}{4} \frac{1}{3} \frac{1}{2\sqrt{3}} \sqrt{3v^2+6v+4} + 6 \frac{1}{3} \frac{1}{2} \sqrt{v\left(\frac{v}{2}+1\right)} \\ &= \frac{4}{3} \sqrt{\frac{3}{2}v^2+3v+2} + \sqrt{v\left(\frac{v}{2}+1\right)} \end{aligned}$$



$$\text{Icosa: } 3\tau^2(\text{Eq.1}) = (3\tau^2 - 4\cos^2\alpha)D^2 - 4\tau^4\cos\alpha D + \tau^4 \equiv F_{11} = 0$$

$$\sin^2\beta_1 = \frac{1}{3\tau^2} \quad 3\tau^2(\text{Eq.2}) = 2[(\cos\alpha - \sqrt{3}\sin\alpha)\cos\alpha - 3\tau^2]D^2 - \tau^4(\cos\alpha + \sqrt{3}\sin\alpha)D + \tau^4 \equiv F_{12} = 0$$

$$F_{13} \equiv F_{12} + yF_{11} = \underbrace{[3\tau^2(y-2) + 2((1-2y)\cos\alpha - \sqrt{3}\sin\alpha)\cos\alpha]}_i D^2 - \underbrace{((4y+1)\cos\alpha + \sqrt{3}\sin\alpha)\tau^4}_j D + \underbrace{(y+1)\tau^4}_k = 0$$

$$\text{Define eta and lambda: } j^2 - 4ik = (\eta\cos\alpha + \lambda\sqrt{3}\sin\alpha)^2 = \eta^2\cos^2\alpha + (\eta\lambda)2\sqrt{3}\cos\alpha\sin\alpha + \lambda^2 3\sin^2\alpha$$

$$= \underbrace{[\tau^2(4y+1)^2 - 4\tau^2(y+1)(3\tau^2(y-2) + 2(1-2y))]}_{\eta^2} \cos^2\alpha + \underbrace{[\tau^8(4y+1) + 4\tau^4(y+1)]}_{(\eta\lambda)} 2\sqrt{3}\cos\alpha\sin\alpha + \underbrace{[\tau^8 - 4\tau^6(y+1)(y-2)]}_{\lambda^2} 3\sin^2\alpha$$

$$(\eta\lambda)^2 - \eta^2 \lambda^2 = [\tau^8(4y+1) + 4\tau^4(y+1)]^2 - [\tau^2(4y+1)^2 - 4\tau^2(y+1)(3\tau^2(y-2) + 2(1-2y))] [\tau^8 - 4\tau^6(y+1)(y-2)] = 0$$

$$=$$

$\tau^{16}$				16 - 16	8 - 8	1 - 1
$\tau^{14}$	64	-32		-144	-80	144
$\tau^{12}$	-48	144	96	0	32 + 128	-288
$\tau^{10}$	64	-32		-192	-32	64
$\tau^8$				16	32	16
$+144\tau^{12}$		$y^4$			$-2y^2$	$-\tau^2 y$

$$\tau^{n-2} + \tau^{n+2} = 3\tau^n$$



First root of  $y$ :  $y_{11} = -1$

Second root:  $p = -1$   $q = -1$   $r = -\tau$

$$a = \frac{-p^2}{3} + q = \frac{-1}{3} - 1 = -\frac{4}{3}$$

$$b = \frac{2p^3}{27} - \frac{pq}{3} + r = -\frac{2}{27} - \frac{1}{3} - \tau = \frac{-49 - 27\sqrt{5}}{54}$$

$$y_{12} = \frac{-p}{3} - \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} - \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} = \phi^2 - \frac{1}{3}$$

$$(y+1) \left[ \begin{array}{r} y^3 - y^2 - y - \tau \\ \hline y^4 - 2y^2 - \tau^2 y - \tau \\ \hline y^4 + y^3 \\ \hline 0 - y^3 - y^2 \\ \hline 0 - y^2 - y \\ \hline 0 - \tau y - \tau \\ \hline 0 \quad 0 \end{array} \right]$$

From the first root of  $y$ :  $\frac{F_{13}}{D} = \frac{F_{12} - F_{11}}{D}$

Eq.5

$$= [2(3\cos\alpha - \sqrt{3}\sin\alpha)\cos\alpha - 9\tau^2]D + \tau^4(3\cos\alpha - \sqrt{3}\sin\alpha) = [2\Gamma\cos\alpha - 9\tau^2]D + \tau^4\Gamma = 0 \text{ so, } D = \frac{\tau^4\Gamma}{9\tau^2 - 2\Gamma\cos\alpha}$$

$$\frac{\Gamma(9\tau^2 - 2\Gamma\cos\alpha)}{3\tau^2 D} F_{11} = \frac{\Gamma(9\tau^2 - 2\Gamma\cos\alpha)}{3\tau^2} \left[ [3\tau^2 - 4\cos^2\alpha] \left( \frac{\tau^4\Gamma}{9\tau^2 - 2\Gamma\cos\alpha} \right) - 4\tau^4\cos\alpha + \tau^4 \left( \frac{9\tau^2 - 2\Gamma\cos\alpha}{\tau^4\Gamma} \right) \right] = 0$$

$$= 4(\Gamma\cos\alpha)^2 - 36\tau^2\Gamma\cos\alpha + 27\tau^2 + \tau^4\Gamma^2 = 0$$

from Eq. 4:  $4(\Gamma\cos\alpha)^2 + 3\tau^2(\tau^4 - 12)\Gamma\cos\alpha + \frac{3}{2}\tau^2(\tau^2 + 18) = \frac{\tau^4}{2}\sqrt{9 - 12\Gamma\cos\alpha(\Gamma\cos\alpha - 3)}$



Square both sides and subtract:

$$16 (\Gamma \cos \alpha)^4 + 24 \tau^2 (\tau^2 - 12) (\Gamma \cos \alpha)^3 + 36 \tau^2 (21 \tau^2 + 11) (\Gamma \cos \alpha)^2 + 54 \tau^4 (\tau^2 - 36) \Gamma \cos \alpha + 81 \tau^4 (\tau^2 + 9) = 0$$

Define  $x$ :  $x \equiv \frac{2}{3} \Gamma \cos \alpha$  and divide by 81:

$$x^4 + \tau^2 (\tau^2 - 12) x^3 + \tau^2 (21 \tau^2 + 11) x^2 + \tau^4 (\tau^2 - 36) x + \tau^4 (\tau^2 + 9) = 0$$

First root of  $x$ :  $x_{11} = 1$

Second root:  $p = -9\tau^2$

$$q = \tau^2 (21\tau^2 + 2) \quad r = -\tau^4 (\tau^2 + 9)$$

$$a = -\frac{p^2}{3} + q = -27\tau^4 + \tau^2 (21\tau^2 + 2) = -2\tau^6$$

$$b = \frac{2p^3}{27} - \frac{pq}{3} + r = -54\tau^6 + 3\tau^4 (21\tau^2 + 2) - \tau^4 (\tau^2 + 9) = \tau^{10}$$

$$x_{12} = -\frac{p}{3} - \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} - \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} = 3\tau^2 - \tau^3 \phi \quad \text{so, } \Gamma \cos \alpha = \frac{3}{2} (3\tau^2 - \tau^3 \phi)$$

$$(x-1) \left[ \begin{array}{r} x^3 \quad -9\tau^2 x^2 + \tau^2 (21\tau^2 + 2)x - \tau^4 (\tau^2 + 9) \\ x^4 + \tau^2 (\tau^2 - 12)x^3 + \tau^2 (21\tau^2 + 11)x^2 + \tau^4 (\tau^2 - 36)x + \tau^4 (\tau^2 + 9) \\ \hline x^4 \quad -x^3 \\ 0 \quad -9\tau^2 x^3 \quad + 9\tau^2 x^2 \\ \hline 0 \quad \tau^2 (21\tau^2 + 2)x^2 - \tau^2 (21\tau^2 + 2)x \\ \hline 0 \quad -\tau^4 (\tau^2 + 9)x + \tau^4 (\tau^2 + 9) \\ \hline 0 \quad 0 \end{array} \right]$$

Eq. 6



note:  $\left( \sqrt[3]{\frac{\tau}{2} + \sqrt{\frac{\tau^2}{4} - \frac{8}{27}}} \right) \left( \sqrt[3]{\frac{\tau}{2} - \sqrt{\frac{\tau^2}{4} - \frac{8}{27}}} \right) = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$  so,  $\phi^3 = 2\phi + \tau$  Eq. 7

also note:  $[9\tau^3 + \tau + 6\phi - 3\tau^3\phi^2]^2 - (\tau + 3\phi)^2 (1 - 3\tau^4(3\sqrt{5} - 2\tau^3\phi + \tau^2\phi^2)) = 0$

$$= (9\tau^3 + \tau)^2 + 36\phi^2 + 9\tau^6\phi(2\phi + \tau) + 2(9\tau^3 + \tau)(6\phi - 3\tau^3\phi^2) - 36\tau^3(2\phi + \tau) - \tau^2(1 - 3\tau^4(3\sqrt{5} - 2\tau^3\phi + \tau^2\phi^2))$$

$$- 6\tau(\phi - 3\tau^4(3\sqrt{5}\phi - 2\tau^3\phi^2 + \tau^2(2\phi + \tau))) - 9(\phi^2 - 3\tau^4(3\sqrt{5}\phi^2 - 2\tau^3(2\phi + \tau) + \tau^2\phi(2\phi + \tau)))$$

$$= (81\tau^6 + 18\tau^4 + \tau^2 - 36\tau^4 - \tau^2 + 9\tau^6\sqrt{5} + 18\tau^8 - 54\tau^8)$$

$$+ (9\tau^7 + 12(9\tau^3 + \tau) - 72\tau^3 + 6\tau^9 - 6\tau + 54\tau^5\sqrt{5} + 36\tau^7 - 108\tau^8 + 27\tau^7)\phi$$

$$+ (36 + 18\tau^6 - 6\tau^3(9\tau^3 + \tau) + 3\tau^8 - 36\tau^8 - 9 + 81\tau^4\sqrt{5} + 54\tau^6)\phi^2 = 0$$

therefore:  $\sqrt{1 - 3\tau^4(3\sqrt{5} - 2\tau^3\phi + \tau^2\phi^2)} = \frac{9\tau^3 + \tau + 6\phi - 3\tau^3\phi^2}{\tau + 3\phi}$  Eq. 8

Eq. 3:

$$\gamma = \frac{-3 + \sqrt{9 - 12\Gamma\cos\alpha(\Gamma\cos\alpha - 3)}}{2\Gamma\cos\alpha} \stackrel{\text{Eq. 6}}{=} \frac{-1 + \sqrt{1 - 3\tau^4(3\sqrt{5} - 2\tau^3\phi + \tau^2\phi^2)}}{3\tau^2 - \tau^3\phi} \stackrel{\text{Eq. 8}}{=} \frac{-1 + \frac{9\tau^3 + \tau + 6\phi - 3\tau^3\phi^2}{\tau + 3\phi}}{3\tau^2 - \tau^3\phi}$$

$$\gamma = \frac{-\tau - 3\phi + 9\tau^3 + \tau + 3\phi + 3(3\tau^2 - \tau^4)\phi - 3\tau^3\phi^2}{(\tau + 3\phi)(3\tau^2 - \tau^3\phi)} = \frac{3\tau + 3\phi}{\tau + 3\phi}$$

$$\cos \alpha = \frac{1}{\sqrt{1 + \tan^2 \alpha}} = \frac{1}{\sqrt{1 + \frac{\gamma^2}{3}}} = \frac{1}{\sqrt{1 + \frac{1}{3} \left( \frac{3\tau + 3\phi}{\tau + 3\phi} \right)^2}} = \frac{\tau + 3\phi}{2\sqrt{\tau^2 + 3\phi(\tau + \phi)}}$$

$$\Gamma = \cos \alpha [3 - \gamma] = \left( \frac{\tau + 3\phi}{2\sqrt{\tau^2 + 3\phi(\tau + \phi)}} \right) \left[ 3 - \frac{3\tau + 3\phi}{\tau + 3\phi} \right] = \frac{3\phi}{\sqrt{\tau^2 + 3\phi(\tau + \phi)}}$$

$$\text{Eq. 5: } D = \frac{\tau^4 \Gamma}{9\tau^2 - 2\Gamma \cos \alpha} \stackrel{\text{Eq. 6}}{=} \frac{\tau \Gamma}{3\phi} = \frac{\tau}{\sqrt{\tau^2 + 3\phi(\tau + \phi)}} \quad r_{\blacktriangle} = \frac{R_{\blacktriangle}}{D} = \frac{\tau}{2\sqrt{3}} \sqrt{\tau^2 + 3\phi(\tau + \phi)}$$

$$r_{\blacklozenge} = \sqrt{r_{\blacktriangle}^2 + (2 \tan 60^\circ)^2 - (2 \tan 36^\circ)^2} = \sqrt{\frac{5\tau^2(\tau^2 + 3\phi(\tau + \phi)) + 5 - 3\tau^3\sqrt{5}}{60}} = \frac{\tau}{2} \sqrt{\frac{1}{\tau\sqrt{5}} + \phi(\tau + \phi)}$$

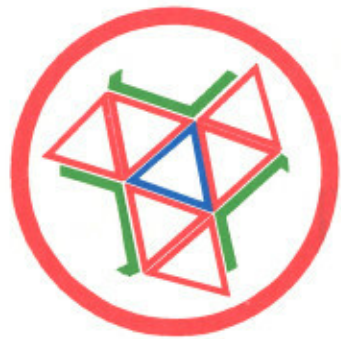
$$\begin{aligned} V_{5\tau^2} &= n_{\blacktriangle} a_{\blacktriangle} \frac{1}{3} r_{\blacktriangle} + n_{\blacklozenge} a_{\blacklozenge} \frac{1}{3} r_{\blacklozenge} = 80 \frac{\sqrt{3}}{4} \frac{1}{3} \frac{\tau}{2\sqrt{3}} \sqrt{\tau^2 + 3\phi(\tau + \phi)} + 12 \frac{5}{4} \sqrt{\frac{\tau^3}{5}} \frac{1}{3} \frac{\tau}{2} \sqrt{\frac{1}{\tau\sqrt{5}} + \phi(\tau + \phi)} \\ &= \frac{10\tau}{3} \sqrt{\tau^2 + 3\phi(\tau + \phi)} + \frac{5\tau^2}{2} \sqrt{\frac{1}{5} + \frac{\tau\phi}{\sqrt{5}}(\tau + \phi)} \end{aligned}$$



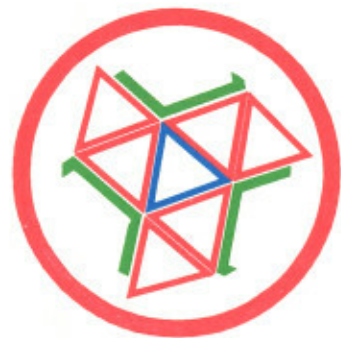
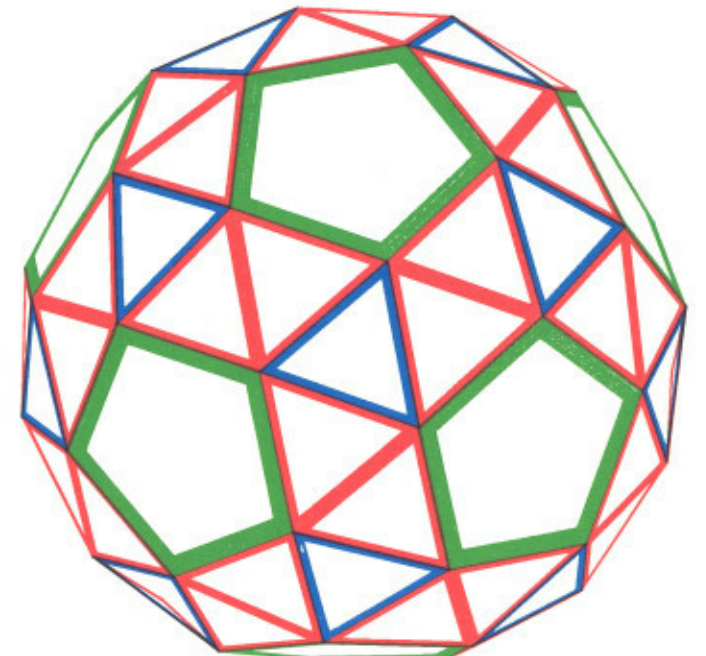
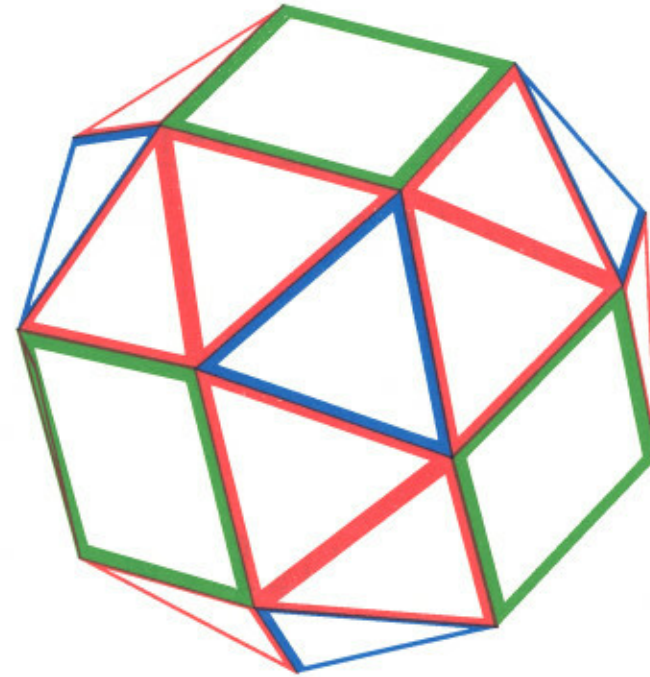
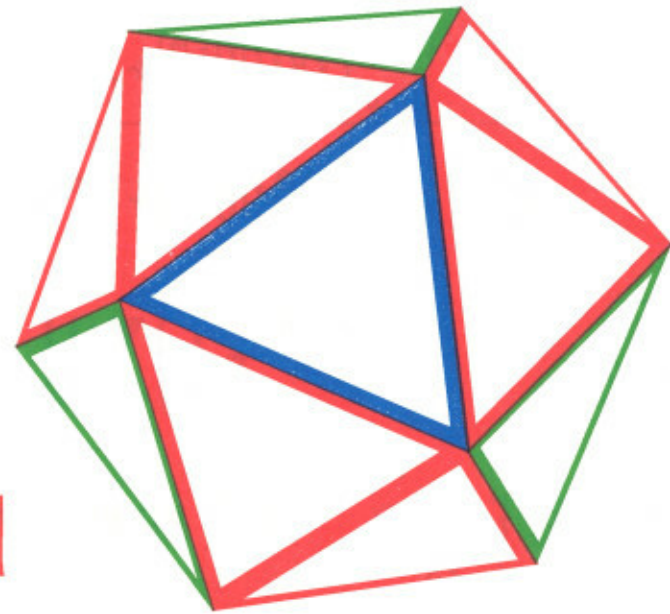
$5_vT$

$2_vA$   $5_vT$

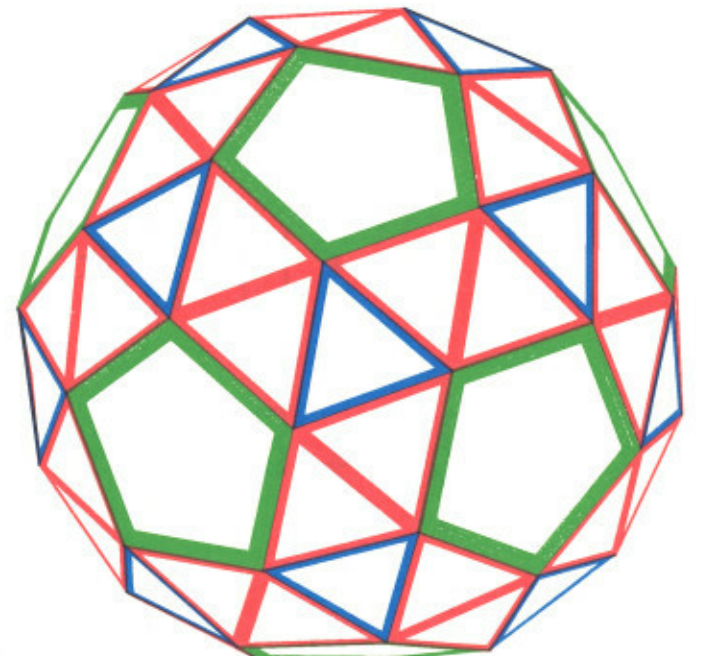
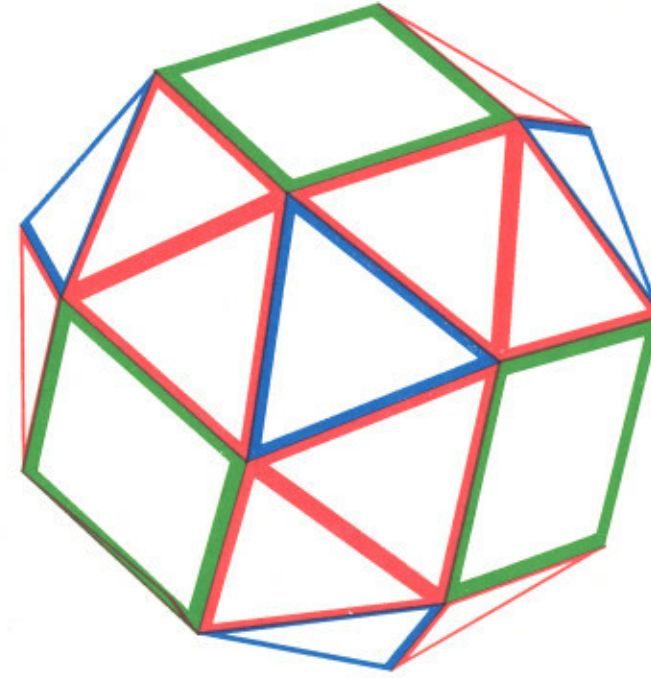
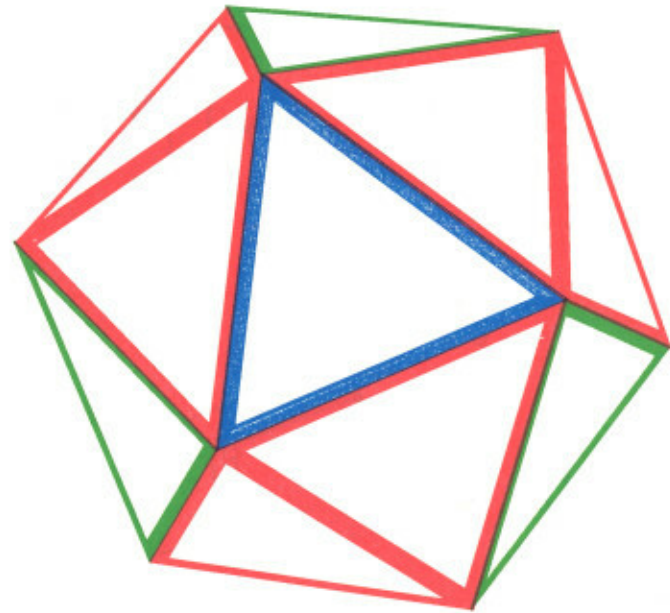
$5_vT^2$



Right - handed



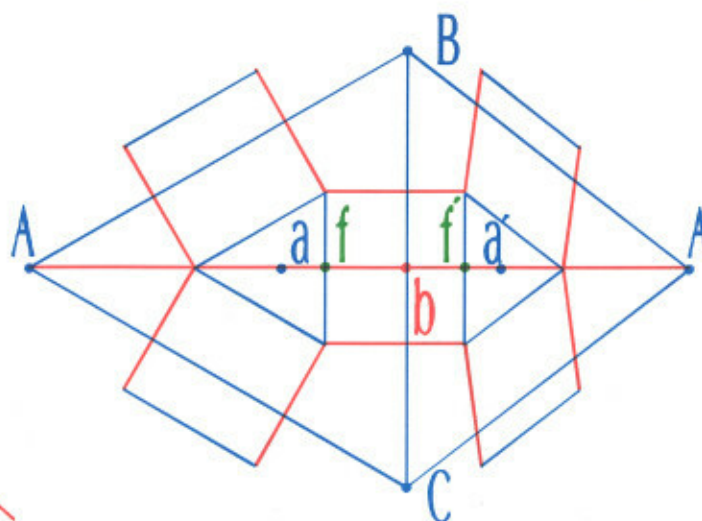
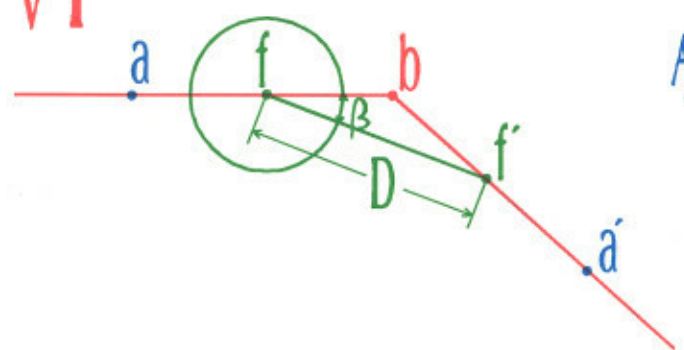
Left - handed







7<sub>v</sub>T



$$\overline{ab} = \overline{af} + \overline{fb}$$

or

$$\frac{1}{2\sqrt{3}} = D \left( \frac{1}{2\sqrt{3}} + \frac{1}{2\cos\beta} \right)$$

$$r_{\blacktriangle} = \frac{R_{\blacktriangle}}{D} = R_{\blacktriangle} \left( 1 + \frac{\sqrt{3}}{\cos\beta} \right)$$

Octa:

$$r_{\blacktriangle} = R_{0\blacktriangle} \left( 1 + \frac{\sqrt{3}}{\cos\beta_0} \right) = \frac{1}{2\sqrt{3}} (3 + \sqrt{2})$$

$$r_{\blacksquare} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60^\circ)^2 - (2\tan 45^\circ)^2} = \sqrt{\frac{1}{12} [11 + 6\sqrt{2} + 1 - 3]} = \frac{1}{2} (1 + \sqrt{2})$$

$$V_{2v\Delta 7vT} = n_{\blacktriangle} a_{\blacktriangle} \frac{1}{3} r_{\blacktriangle} + n_{\blacksquare} a_{\blacksquare} \frac{1}{3} r_{\blacksquare} = 8 \frac{\sqrt{3}}{4} \frac{1}{3} \frac{1}{2\sqrt{3}} (3 + \sqrt{2}) + 18 \frac{1}{3} \frac{1}{2} (1 + \sqrt{2}) = \frac{2}{3} (6 + 5\sqrt{2})$$

Icosa:

$$r_{\blacktriangle} = R_{1\blacktriangle} \left( 1 + \frac{\sqrt{3}}{\cos\beta_1} \right) = \frac{3 + 2\sqrt{5}}{2\sqrt{3}}$$

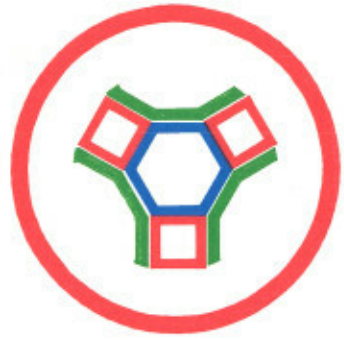
$$r_{\blacksquare} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60^\circ)^2 - (2\tan 45^\circ)^2} = \sqrt{\frac{1}{12} [(29 + 12\sqrt{5}) + 1 - 3]} = \frac{T^3}{2}$$

$$r_{\blacklozenge} = \sqrt{r_{\blacktriangle}^2 + (2\tan 60^\circ)^2 - (2\tan 36^\circ)^2} = \sqrt{\frac{1}{60} [5(29 + 12\sqrt{5}) + 5 - 15 - 6\sqrt{5}]} = \frac{3}{2} \sqrt{\frac{T^3}{5}}$$

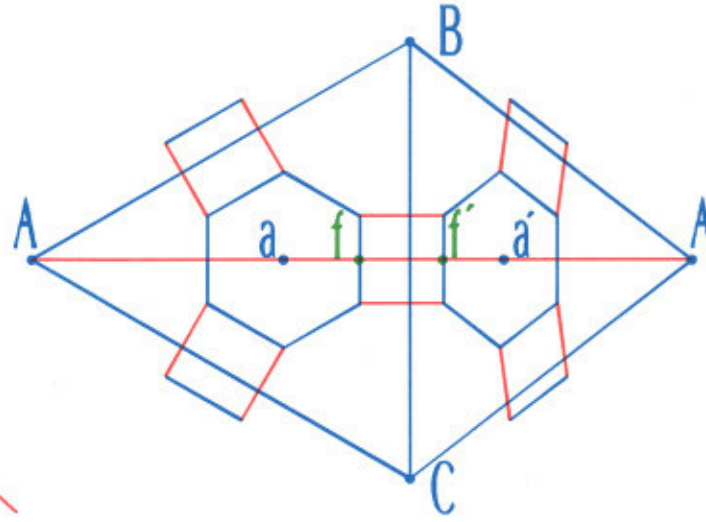
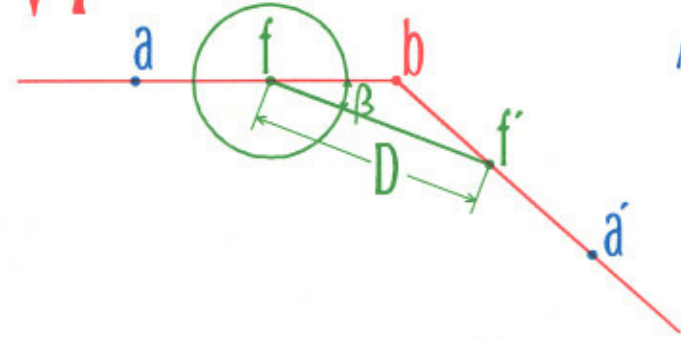
$$V_{5v7vT} = n_{\blacktriangle} a_{\blacktriangle} \frac{1}{3} r_{\blacktriangle} + n_{\blacksquare} a_{\blacksquare} \frac{1}{3} r_{\blacksquare} + n_{\blacklozenge} a_{\blacklozenge} \frac{1}{3} r_{\blacklozenge} = 20 \frac{\sqrt{3}}{4} \frac{1}{3} \frac{3 + 2\sqrt{5}}{2\sqrt{3}} + 30 \frac{1}{3} \frac{2 + \sqrt{5}}{2} + 12 \frac{5}{4} \sqrt{\frac{T^3}{5}} \frac{1}{3} \frac{3}{2} \sqrt{\frac{T^3}{5}}$$

$$= \frac{1}{3} (60 + 29\sqrt{5})$$





$9vT$



$$\overline{ab} = \overline{af} + \overline{fb}$$

or

$$\frac{1}{2\sqrt{3}} = D \left( \frac{\sqrt{3}}{2} + \frac{1}{2\cos\beta} \right)$$

$$r_{\bullet} = \frac{R_{\blacktriangle}}{D} = R_{\blacktriangle} \left( 3 + \frac{\sqrt{3}}{\cos\beta} \right)$$

Octa:  $r_{\bullet} = R_{\blacktriangle}(3 + \frac{\sqrt{3}}{\cos\beta_0}) = \frac{\sqrt{3}}{2}(1 + \sqrt{2})$

$$r_{\blacksquare} = \sqrt{r_{\bullet}^2 + (2\tan 30^\circ)^2 - (2\tan 45^\circ)^2} = \sqrt{\frac{1}{4}[3(3+2\sqrt{2}) + 3 - 1]} = \frac{1}{2}(3 + \sqrt{2})$$

$$r_{\bullet} = \sqrt{r_{\bullet}^2 + (2\tan 30^\circ)^2 - (2\tan 22\frac{1}{2}^\circ)^2} = \sqrt{\frac{1}{4}[3(3+2\sqrt{2}) + 3 - 3 - 2\sqrt{2}]} = \frac{1}{2}(1 + 2\sqrt{2})$$

$$V_{2v\blacktriangle 9vT} = n_{\bullet}a_{\bullet}\frac{1}{3}r_{\bullet} + n_{\blacksquare}a_{\blacksquare}\frac{1}{3}r_{\blacksquare} + n_{\bullet}a_{\bullet}\frac{1}{3}r_{\bullet} = 8\frac{3\sqrt{3}}{2}\frac{1}{3}\frac{\sqrt{3}}{2}(1+\sqrt{2}) + 12\frac{1}{3}\frac{1}{2}(3+\sqrt{2}) + 6\frac{2(1+\sqrt{2})}{3}\frac{1}{2}(1+2\sqrt{2}) = 30 + 14\sqrt{2}$$

Icosa:  $r_{\bullet} = R_{\blacktriangle}(3 + \frac{\sqrt{3}}{\cos\beta_1}) = \frac{\tau^3\sqrt{3}}{2}$

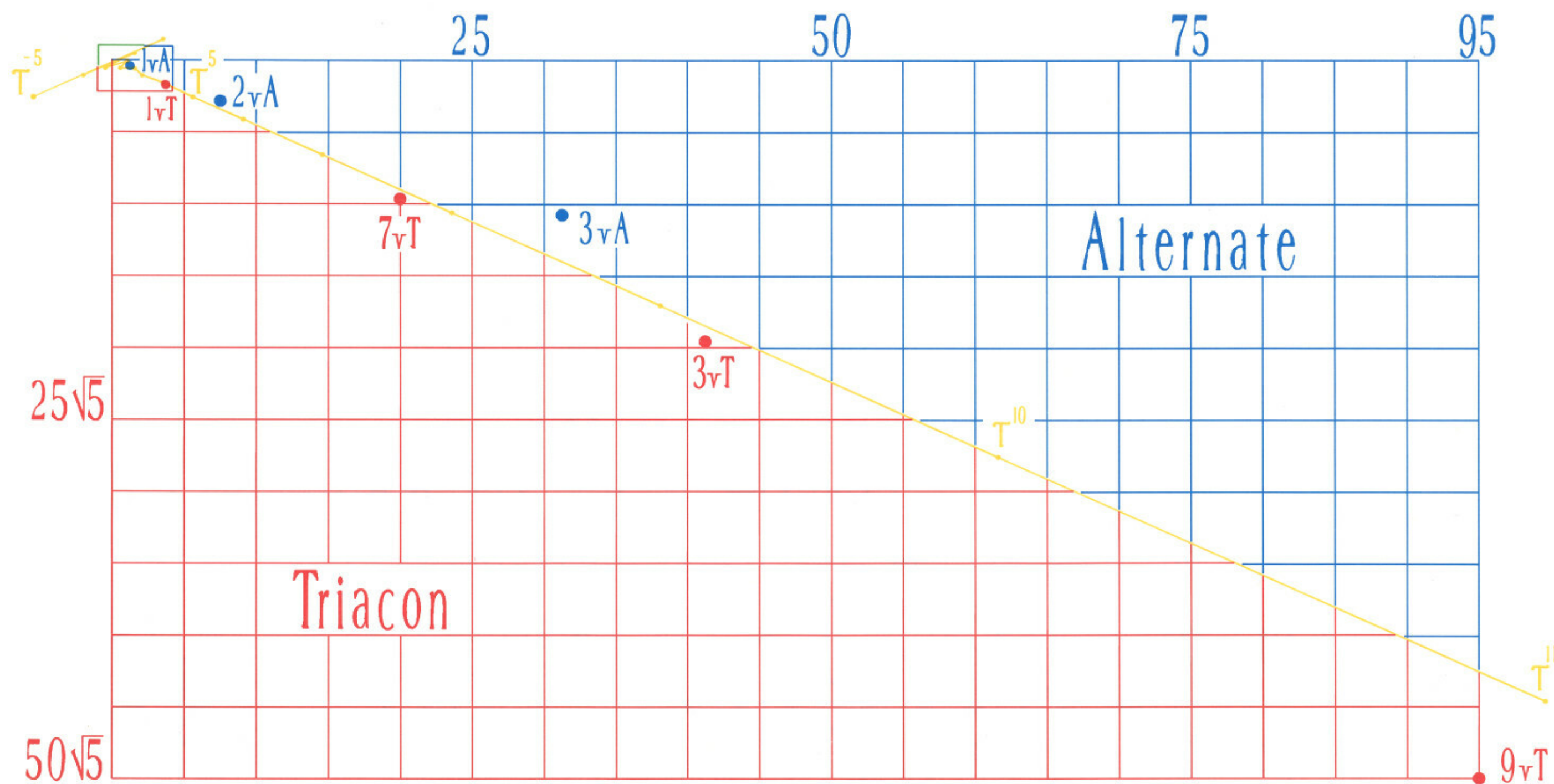
$$r_{\blacksquare} = \sqrt{r_{\bullet}^2 + (2\tan 30^\circ)^2 - (2\tan 45^\circ)^2} = \sqrt{\frac{1}{4}[3(9+4\sqrt{5}) + 3 - 1]} = \frac{1}{2}(3 + 2\sqrt{5})$$

$$r_{\bullet} = \sqrt{r_{\bullet}^2 + (2\tan 30^\circ)^2 - (2\tan 18^\circ)^2} = \sqrt{\frac{1}{4}[3(9+4\sqrt{5}) + 3 - 5 - 2\sqrt{5}]} = \frac{1}{2}\sqrt{5}\tau^3\sqrt{5}$$

$$V_{5v9vT} = n_{\bullet}a_{\bullet}\frac{1}{3}r_{\bullet} + n_{\blacksquare}a_{\blacksquare}\frac{1}{3}r_{\blacksquare} + n_{\bullet}a_{\bullet}\frac{1}{3}r_{\bullet} = 20\frac{3\sqrt{3}}{2}\frac{1}{3}\frac{\sqrt{3}}{2}\tau^3 + 30\frac{1}{3}\frac{1}{2}(3+2\sqrt{5}) + 12\frac{5\sqrt{\tau^3\sqrt{5}}}{2}\frac{1}{2}\sqrt{5}\tau^3\sqrt{5} = 95 + 50\sqrt{5}$$

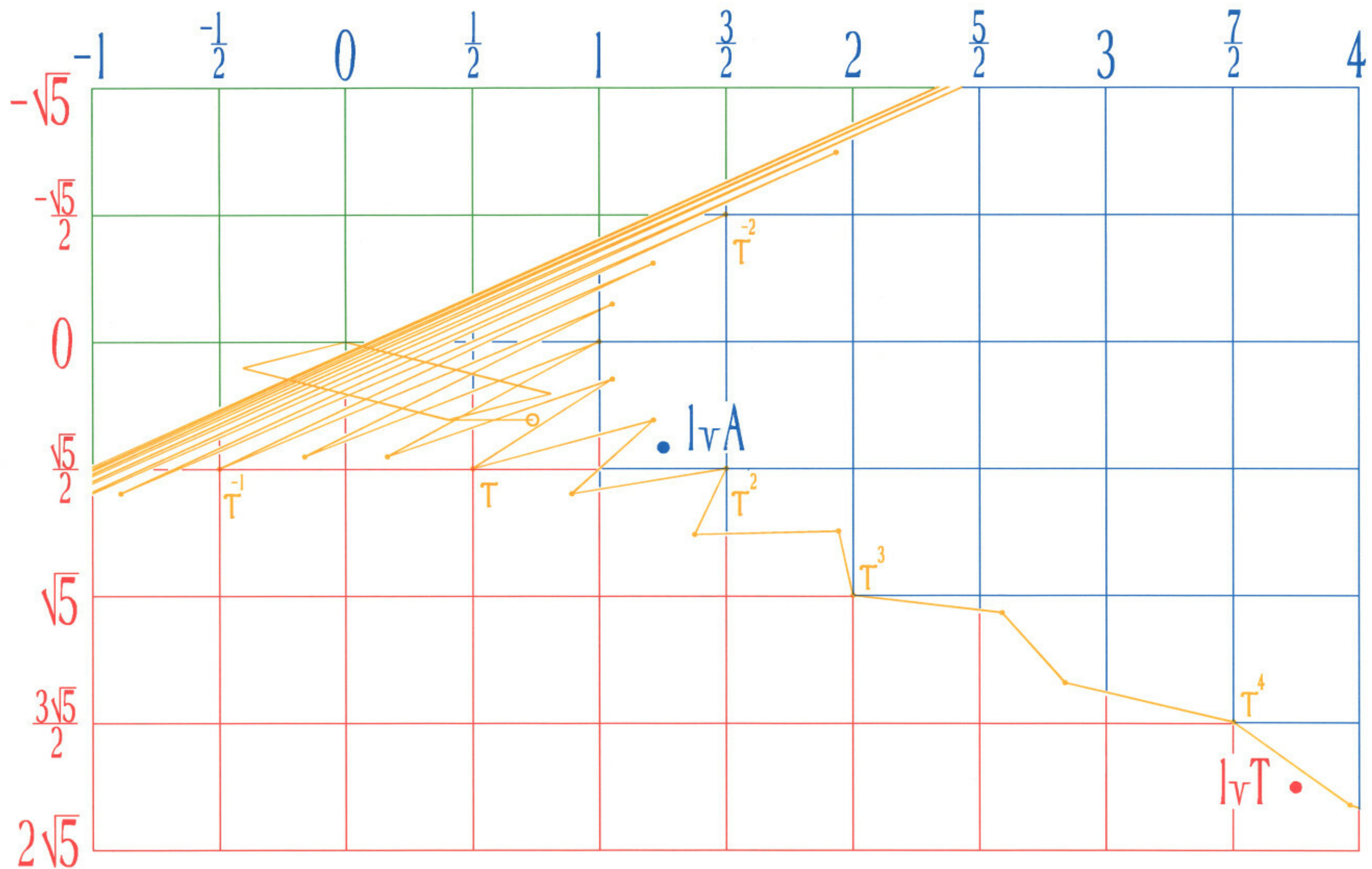
# Integer powers of the Golden Section

## Plotted volumes of icosahedral based solids





# One third powers of the Golden Section



## Exercise:

Spinnability of the first golden circle leads to a new second root.  
Plot this root.

## Bibliography:

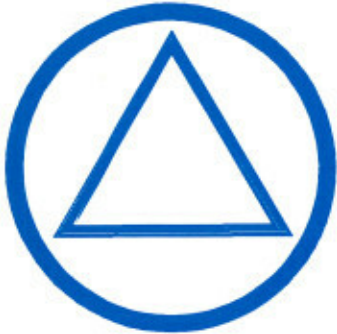




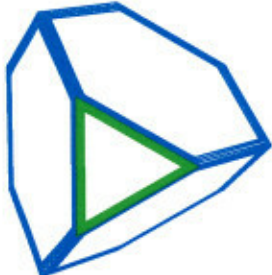
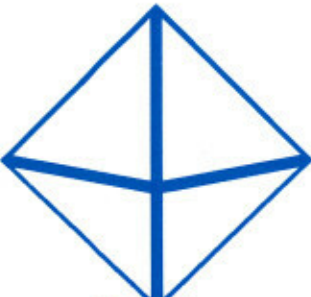
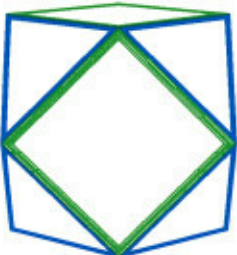




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# Alternate and

# Triacon Breakdowns

1v	2v	3v
		
		
tetra		
		
octa		
		
icosa		

1v	3v	5v	7v	9v
